

# Numerical Solution of the Spinor-Spinor Bethe-Salpeter Equation in Functional Nonlinear Spinor Theory

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The mass eigenvalue equation for mesons in nonlinear spinor theory is derived by functional methods. In second order it leads to a spinorial Bethe-Salpeter equation. This is solved by a variational method with high precision for arbitrary angular momentum. The results for scalar mesons show a shift of the first order results, obtained earlier. The agreement with experiment is improved thereby. An excited state corresponding to the  $\eta'$  is found. A calculation of a Regge trajectory is included, too.

## 1. Introduction

Nonlinear spinor theory is an attempt for a unified description of high energy phenomena<sup>1</sup>. In its original formulation by Heisenberg, it was defined as an operator field theory. The meaning of these operators, however, is not at all clear, since they act not in a Hilbert space but in a space with indefinite metric. So they do not possess a simple particle interpretation. Therefore, in the field equation only the underlying symmetries of the theory are stated, but it is not suited for performing dynamical calculations.

Such calculations are only possible with the help of the  $\tau$ -function system and similar functions which are derived from these. The  $\tau$ -functions are the transition matrix elements of time-ordered products of field operators between the vacuum and an arbitrary state. They can be described in compact fashion by a generating functional. This is well-known from other models of high energy physics<sup>2</sup>. Usually they are considered only as an appropriate tool for performing complicated manipulations. It was the proposal of Stumpf<sup>3</sup> to work only with those functionals and to give up the connection with usual operator field theory.

The aim of this program is therefore to develop a self-contained functional quantum theory. Besides the definition of the symmetries it should allow the calculation of bound and scattering states. In this paper we deal only with the bound state problem. The solution of this task is of major importance also for the scattering problem, because it will supply us with the wave-functions of the in- and outgoing particles.

In nonlinear spinor theory all particles are treated as composite. Therefore the usual LSZ reduction

technique cannot be used for the calculation of the S-matrix. The functional quantum theory provides methods for calculating cross-sections of such composite objects. In fact the possibility for doing such calculations was the main motivation for their introduction. We will not go into the details here but refer to the literature<sup>4</sup>.

Even if all particles should be considered on the same footing in nonlinear spinor theory, the nucleon has a somewhat distinguished position, as the propagator of the fundamental field  $\psi(x)$  has a physical pole at the nucleon mass. In this sense we will call  $\psi(x)$  the nucleon field. This should not mean that  $\psi(x)$  is a local interpolating nucleon field in the usual LSZ-sense. In fact, it cannot be such a field since  $\psi(x)$  has a noncanonical dimension.

Besides the nucleon pole, the propagator has a double pole at mass zero, corresponding to a dipole ghost. We consider it as pure means for regularizing the theory without meaning as a physical particle. Therefore we do not follow the attempts to bring it into connection with the leptons<sup>5</sup>. We consider nonlinear spinor theory purely as a model of strong interactions and do not try to establish weak and electromagnetic interactions in its framework.

Next to the nucleon, the mesons (and the deuteron) are the most simple system. They are considered as a bound state of the nucleon and the antinucleon. With their properties, especially with the calculation of their masses, we will deal in this paper. The masses are found as eigenvalues of Bethe-Salpeter type equations (for a review of the theory of the Bethe-Salpeter equation see<sup>6</sup>). The numerical solution of these equations is the central problem we have to tackle in the course of this paper.



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In lowest order approximation, the so-called contact graph, meson masses have been calculated in nonlinear spinor theory already in the past<sup>1,7</sup>. It is a purely algebraic equation because of the contact type interaction. Therefore it yields solutions only for systems without orbital angular momentum. The eigenvalues for scalar mesons ( $\pi$  and  $\eta$ ) are of the correct order of magnitude. Therefore it is highly desirable to investigate the second order in order to see its effect on the first order result.

The second order approximation consists of a nonlocal graph corresponding to a particle exchange. Its solution is much more complicated. Up to now, it has only been investigated in first Fredholm approximation. Even these calculations were incomplete: In<sup>8</sup> no angular momentum reduction has been performed. This was done in<sup>9</sup>, but there parts of the amplitudes have been neglected without motivation.

The first complete calculation of the first Fredholm approximation has been done in<sup>10</sup>. Its result was quite disastrous: There was no more any solution for the  $\pi$  or the  $\eta$  meson, and the first order results turned out to be small corrections to the second order thus invalidating the whole approximation scheme.

Investigations on the scalar Bethe-Salpeter equation<sup>11</sup>, however, showed that this result was inconclusive. The Fredholm approximation was found to yield wrong results, differing from the exact ones by factors of 2 or 3. Therefore it should be used no longer as a solution procedure. For the solution of the second order equation one has to use therefore a numerical method. The numerical solution of spinorial Bethe-Salpeter equation has not been considered very much in the literature so far. Numerous results are only known for a harmonic interaction<sup>12,13</sup>. Compared to the general case, matters are simpler in nonlinear spinar theory for two reasons:

1) Because of the indefinite metric, the interaction kernels are automatically regularized, having no singularities on the light-cone. Thus all well-known difficulties with marginal singular potentials<sup>14</sup> are circumvented.

2) There is no mass term in the fundamental field equation. Therefore the theory will be chiral invariant (unless spontaneously broken). This manifests itself in the decoupling of the Bethe-Salpeter equation into two identical sets, thus re-

ducing the number of the coupled equations by a factor of two.

The variational method has been found to be the most effective numerical method for solving the scalar Bethe-Salpeter equation<sup>15</sup>. Therefore we will use it for the spinar case, too. In doing so we rely on the work of Ladányi<sup>16,17</sup>. In these papers the spinorial Bethe-Salpeter equation has been put into a form suitable for the application of the variational method. For a numerical computation it has not been used up to now. All previous papers<sup>18-20</sup> transform the integral equation in momentum space into a matrix equation which is solved by inversion. The variational method, however, will turn out to be of equal strength.

The results of our work concerning nonlinear spinor theory are twofold: Firstly, we find for scalar mesons that the second order is only a small correction. So the hope is affirmed, that the  $\pi$  and the  $\eta$  meson can be considered as true bound states of the theory and not only as an effect of the first order approximation. Secondly, we now get bound states with arbitrary spin. But they appear only for coupling constants of much higher value than used in the past. As an example, we will calculate a Regge trajectory.

## 2. Derivation of the Dynamical Equation

We start with the derivation of the dynamical equation from which the meson masses shall be calculated. According to the philosophy stated in the introduction this will be done with the tools provided by functional quantum theory. The presentation will be quite short, for details we refer to the literature, see e. g.<sup>3</sup>.

The spinor field equation, formulated for the generating functionals  $|\mathcal{Z}_a(j)\rangle$  reads

$$\{D_{\alpha\beta}\partial_\beta - V_{\alpha\beta\gamma\delta}\partial_\beta\partial_\gamma\partial_\delta\}|\mathcal{Z}_a(j)\rangle = 0. \quad (2.1)$$

We use the Hermitean Dirac representation, the relation to the Weyl representation, used in most of the earlier papers on nonlinear spinor theory, will be discussed in the next section. Hence every index in (2.1) stands for a collection of three discrete indices  $a_1 a_2 a_3$  and a continuous variable  $x$ . A summation, resp. integration over repeated indices is always implied.

The meaning of the different symbols in (2.1) is as follows:  $|\mathcal{Z}_a(j)\rangle$  is the generating functional of

the  $\tau$ -functions of the state  $|a\rangle$ :

$$|\mathcal{T}_a(j)\rangle = \sum_{n=0}^{\infty} \frac{i^n}{n!} \tau_n(\alpha_1, \dots, \alpha_n) j_{\alpha_1} \dots j_{\alpha_n} |\varphi_0\rangle. \quad (2.2)$$

$D_{\alpha\beta}$  is the usual Dirac differential operator for mass zero particles:

$$D_{\alpha\beta} = -i(\lambda_2' \beta \gamma^\nu)_{\alpha\beta} \partial / \partial x_\beta^\nu \quad (2.3)$$

(the  $\delta$ -function occurs because of the summation convention).  $V_{\alpha\beta\gamma\delta}$  is the totally antisymmetric local vertex operator. In the Dirac formulation it is no longer uniquely determined by the requirements of Lorentz invariance and Fierz antisymmetry as in the Weyl case. Additionally we have to demand that it reduces to a local vertex also in the Weyl form. Explicitly, it reads<sup>21</sup>:

$$\begin{aligned} V_{\alpha\beta\gamma\delta} &= \sum_{i=1}^8 V_{\alpha\beta}^{(i)} V_{\gamma\delta}^{(i)} \\ &= \frac{1}{24} l^2 \delta(x_\alpha - x_\beta) \delta(x_\alpha - x_\gamma) \delta(x_\alpha - x_\delta) \bar{V}_{\alpha\beta\gamma\delta}, \\ \bar{V}_{\alpha\beta\gamma\delta} &= 3(\beta \gamma'^\mu)_{\alpha\beta} (\beta \gamma'_\mu)_{\gamma\delta} \\ &\quad + 3(\beta \gamma'_5 \gamma'^\mu)_{\alpha\beta} (\beta \gamma'_5 \gamma'_\mu)_{\gamma\delta} \\ &\quad + (\lambda_2' \tau_i' \beta \gamma'^\mu)_{\alpha\beta} (\lambda_2' \tau_i' \beta \gamma'_\mu)_{\gamma\delta} \\ &\quad + (\lambda_2' \tau_i' \beta \gamma'_5 \gamma'^\mu)_{\alpha\beta} (\lambda_2' \tau_i' \beta \gamma'_5 \gamma'_\mu)_{\gamma\delta} \\ &\quad + 2(\lambda_1' \tau_i' C)_{\alpha\beta} (\lambda_1' \tau_i' C)_{\gamma\delta} \\ &\quad + 2(\lambda_1' \tau_i' \gamma'_5 C)_{\alpha\beta} (\lambda_1' \tau_i' \gamma'_5 C)_{\gamma\delta} \\ &\quad + 2(\lambda_3' \tau_i' C)_{\alpha\beta} (\lambda_3' \tau_i' C)_{\gamma\delta} \\ &\quad + 2(\lambda_3' \tau_i' \gamma'_5 C)_{\alpha\beta} (\lambda_3' \tau_i' \gamma'_5 C)_{\gamma\delta} \end{aligned} \quad (2.4)$$

(the corresponding expression in<sup>7</sup> contains some mistakes). All matrices in (2.4) are defined as direct products in  $\lambda$ -, isospin- and spin-space:

$$\begin{aligned} \lambda' &= (\lambda_1 c_\tau, -i \lambda_2, \lambda_3 c_c), \\ \tau' &= (-i \lambda_2 \tau_1, i \tau_2, -i \lambda_2 \tau_3), \\ \gamma'^\mu &= (-i \lambda_2 \gamma^0, -i \lambda_2 \gamma^1, i \gamma^2, -i \lambda_2 \gamma^3). \end{aligned} \quad (2.5)$$

In (2.1) we have omitted a commutator term  $\sim \varrho_0$  which usually occurs in the equations for generating functionals: Due to the assumed indefinite metric in the state space,  $\varrho_0 = 0$  and the commutator term is absent. This is of great importance since it prevents to derive a Bethe-Salpeter equation from (2.1) by the conventional procedure, starting from the inhomogeneous equation.

Therefore we will proceed differently, as indicated in<sup>22</sup>. First we invert the differential operator in (2.1) with the causal Green's function  $G_{\alpha\beta}$ :

$$D_{\alpha\beta} G_{\beta\gamma} = \delta_{\alpha\gamma} \quad (2.6)$$

and transform to the  $|\Phi_a(j)\rangle$ -functional:

$$|\mathcal{T}_a(j)\rangle = \exp \left\{ -\frac{1}{2} j_\alpha F_{\alpha\beta} j_\beta \right\} |\Phi_a(j)\rangle. \quad (2.7)$$

$F_{\alpha\beta}$  is the (suitably regularized) two-point function of the theory. We will discuss its explicit form in the next section. With the definition

$$d_\alpha = \partial_\alpha - F_{\alpha\beta} j_\beta \quad (2.8)$$

we obtain the following equation for  $|\Phi_a(j)\rangle$ :

$$\begin{aligned} \partial_\alpha |\Phi_a(j)\rangle &= \{F_{\alpha\beta} j_\beta + G_{\alpha\beta} V_{\beta\gamma\delta\epsilon} d_\gamma d_\delta d_\epsilon\} |\Phi_a(j)\rangle \\ &=: O_\alpha |\Phi_a(j)\rangle. \end{aligned} \quad (2.9)$$

In order to remove the asymmetry with respect to one coordinate in (2.9), we have to symmetrize (2.9). As we shall see below, this is not without problems. There have been proposed essentially two different symmetrization procedures. The one proposed in<sup>22</sup> and employed in<sup>9,10</sup> uses the subsidiary condition for the momentum of the state  $|a\rangle$ . However, here we will follow<sup>23</sup> since this procedure yields equations of more conventional form. It makes use of an indefinite functional integration defined by:

$$\begin{aligned} \int \delta j_\alpha |\mathcal{T}_a(j)\rangle &= |\mathcal{G}(j)\rangle: \\ \Leftrightarrow \partial_\alpha |\mathcal{G}(j)\rangle &= |\mathcal{T}_a(j)\rangle. \end{aligned} \quad (2.10)$$

Applying this to (2.9) yields the symmetrical equation:

$$|\Phi_a(j)\rangle = \int \delta j_\alpha O_\alpha |\Phi_a(j)\rangle =: \mathcal{E} |\Phi_a(j)\rangle. \quad (2.11)$$

Because of the quantum numbers of the state  $|a\rangle$  there will be a lowest  $\varphi$ -function  $\varphi_\varrho$ . From (2.7) we find  $\varphi_\varrho = \tau_\varrho$ ,  $\tau_\varrho$  being the lowest  $\tau$ -function. Defining the projection operator

$$\begin{aligned} \mathbf{P}_\varrho &= \frac{1}{\varrho!} j_{\alpha_1} \dots j_{\alpha_\varrho} |\varphi_0\rangle \langle \varphi_0| \partial_{\alpha_\varrho} \dots \partial_{\alpha_1}, \\ \prod_{\varrho=0}^{\infty} \mathbf{P}_\varrho & \end{aligned} \quad (2.12)$$

and the functional

$$\begin{aligned} |\mathcal{T}_\varrho(j)\rangle &= \frac{i^\varrho}{\varrho!} \tau_\varrho(\alpha_1 \dots \alpha_\varrho) j_{\alpha_1} \dots j_{\alpha_\varrho} |\varphi_0\rangle \\ &= |\Phi_\varrho(j)\rangle \end{aligned} \quad (2.13)$$

with

$$\begin{aligned} \mathbf{P}_\varrho |\mathcal{T}_a(j)\rangle &= |\mathcal{T}_\varrho(j)\rangle, \\ \mathbf{P}_\varrho |\Phi_a(j)\rangle &= |\Phi_\varrho(j)\rangle, \end{aligned} \quad (2.14)$$

we can derive an equation for the lowest functional  $|\mathcal{T}_\varrho(j)\rangle$ :

$$|\mathcal{T}_\varrho(j)\rangle = \mathbf{P}_\varrho \mathcal{E} [1 - \prod_{\varrho=0} \mathcal{E}]^{-1} |\mathcal{T}_\varrho(j)\rangle. \quad (2.15)$$

The inverse operator in (2.15) is now expanded into a Neumann series. This corresponds to the perturbative expansion of the interaction kernel in the conventional Bethe-Salpeter equation. We will keep only the two first terms of this series. For mesons on which we will concentrate from now on, we have  $Q = 2$ . So we have the desired dynamical equation:

$$|\mathfrak{L}_2(j)\rangle = \mathbf{P}_2 \mathfrak{G}(1 + \prod_2 \mathfrak{G}) |\mathfrak{L}_2(j)\rangle. \quad (2.16)$$

For practical purposes the equation has to be transformed to coordinate space. This is achieved simply by applying  $\langle \varphi_0 | \partial_\alpha \partial_\beta$  to it. The result will be written down in a graphical notation which is explained e. g. in <sup>7</sup>. It reads

$$\begin{aligned} & \text{Diagram 1} \dots -\frac{3}{4} \left( \text{Diagram 2} + \text{Diagram 3} \right) - \frac{9}{2} \text{Diagram 4} \\ & -\frac{9}{2} \left( \text{Diagram 5} + \text{Diagram 6} \right) \\ & -\frac{9}{4} \left( \text{Diagram 7} + \text{Diagram 8} \right) \end{aligned} \quad (2.17)$$

If we take into account only the first term in (2.16) we get the first order equation considered before<sup>1</sup>:

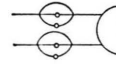
$$\text{Diagram 1} \dots -\frac{3}{2} \left( \text{Diagram 2} + \text{Diagram 3} \right) \quad (2.18)$$

Hence in the second order, the coefficient of the contact graph (2.18) is changed as has been already observed in <sup>24</sup>. This happens as the first term in  $O_\alpha$  (2.9) has no coupling constant. So all graphs arising from it are of lower order in the coupling constant than those generated by the second term. In higher orders of the Neumann series, the coefficients of the second order graphs will therefore be changed, too.

Compared to the conventional Bethe-Salpeter equation, (2.17) shows two distinctive features: First it contains not only the exact propagator  $F$ , but also the undressed Green's function  $G$ . Secondly, the last row of (2.17) contains self-energy graphs which are not two particle irreducible as it is required for graphs showing up in the interaction kernel.

These two deficiencies are intimately connected: The  $G$ -lines will be dressed to  $F$ -lines by the self-energy graphs as it has been discussed at length in <sup>25</sup> in a somewhat modified theory. There it is shown that one gets a symmetrical equation without performing the symmetrization on (2.9). Unfortunately, we cannot take over these methods to our case since we have no Schwinger-Dyson equation for the propagator  $F$  because of the indefinite metric state space.

But we can comment on the combinatorics of (2.17). By the symmetrization, the coefficient of the self-energy graph is changed from 3 to 9/4 so that the  $G$ -lines are no longer correctly dressed. In the same way, the quadratical self-energy terms found in fourth order:



get the coefficient  $\frac{27}{4}$  instead of  $\frac{9}{4} \cdot \frac{9}{4} = \frac{81}{16}$ . Furthermore,  $F$ -lines will be dressed a second time by the symmetrization, thus producing unwanted double poles. Summarizing, we find that the symmetrization destroys the correct combinatoric. But since it is the only mean for producing an order by order anti-symmetrical equation, we have no substitute for it. So we will avoid the self-energy problem by simply neglecting the corresponding graphs.

But this will not be the only simplification of (2.17). We shall also neglect the second row since these graphs are of a similar topological structure as the other nonlocal graph. And for the moment we will disregard the first order term which is well-known. Its contribution will be taken into account in section 7. So we are left with the rather simple looking equation:

$$\text{Diagram 1} \dots -\frac{9}{2} \text{Diagram 4} \quad (2.19)$$

We will devote the next four sections to its solution.

### 3. Group-theoretical Reduction

Since we want to use a variational method for the solution of (2.19) following <sup>16</sup> we first transform it to a differential equation. If we introduce the righthand side differential operator  $D_{\alpha\beta}$  by

$$\tau_{\alpha\beta} \bar{D}_{\beta\gamma} = -D_{\beta\gamma} \tau_{\alpha\beta} \quad (3.1)$$

$$(2.19) \text{ reads: } D_{\alpha\beta} \tau_{\beta\gamma} \bar{D}_{\gamma\delta} = \frac{9}{2} \sum_{i,j=1}^8 V_{\alpha\alpha'}^{(i)} F_{\alpha'\alpha''} V_{\alpha''\beta}^{(j)} \tau_{\beta\gamma} V_{\gamma\gamma'}^{(i)} F_{\gamma'\gamma''} V_{\gamma''\delta}^{(j)}. \quad (3.2)$$



Because of the complicated form of the vertex operator in the Hermitean representation (2.4) the explicit expression is rather long and not very enlightening. So we will not write it down here.

But a considerable simplification arises by performing the reduction with respect to the inner symmetries of the theory. They are given by the baryon number gauge group and the isospin group. The reduction with respect to these groups has been described in detail in <sup>9</sup>, so we will not repeat the technicalities here. Since we are interested in the meson masses, we have the baryon number  $B=0$ , and the possible values for the isospin are  $T=0$  and  $T=1$ . Additionally, we could get an equation for baryon number  $B=2$ , i. e. for the deuteron, but we will not consider it here.

Before writing down the result of these calculations, we have to specify the form of the propagator  $F$ . By the usual requirements of invariance with respect to Poincaré transformation including reflection, its form is restricted to:

$$F_{a\beta}(x, y) = \int \frac{d^4 p}{(2\pi)^4} \exp \{i p \cdot (x - y)\} \int dm^2 \left[ -i \lambda_2' \beta \gamma'^\mu p_\mu \frac{\varrho_1(m^2)}{p^2 - m^2 + i\epsilon} - i \lambda_2' \beta m \frac{\varrho_2(m^2)}{p^2 - m^2 + i\epsilon} \right] \quad (3.3)$$

But (2.1) is, in addition, invariant with respect to chiral transformations because of the absence of a mass term in (2.3). So the propagator should exhibit this symmetry property, too, if it is not broken spontaneously by the vacuum. This possibility will not be considered here, so we conclude:

$$\varrho_2(m^2) \equiv 0. \quad (3.4)$$

In that case, the propagator can be written in momentum space:

$$F(p) = \int dm^2 \frac{1}{2} \varrho_1(m^2) (-i \lambda_2' \beta) \left[ \frac{\gamma'^\mu p_\mu - m}{p^2 - m^2 + i\epsilon} + \frac{\gamma'^\mu p_\mu + m}{p^2 - m^2 + i\epsilon} \right]. \quad (3.5)$$

From this form it is seen that it corresponds to the superposition of propagators for a couple of particles with opposite parity. Hence we have the clearly unphysical situation of a parity-degenerate nucleon doublet. Correspondingly, also the mesonic spectrum will show this unrealistic behaviour. In future papers the possibility of breaking the symmetry should be considered seriously.

The spectral function  $\varrho_1(m^2)$  is assumed to be regularized because of the indefinite metric of the state space. We use the form, conventional in nonlinear spinor theory <sup>1</sup>:

$$\varrho_1(m^2) = \delta(m^2 - \kappa^2) - \delta(m^2) + \kappa^2 \delta'(m^2) \quad (3.6)$$

$\kappa$  being the nucleon mass. So the propagator reads finally:

$$F_{a\beta}(x, y) = -(\lambda_2' \beta \gamma'^\mu)_{a\beta} F_\mu(x - y) \\ F_\mu(x - y) = \int \frac{d^4 p}{(2\pi)^4} \exp \{i p \cdot (x - y)\} \frac{i \kappa^4 p_\mu}{(p^2 + i\epsilon)^2 (p^2 - \kappa^2 + i\epsilon)}. \quad (3.7)$$

With these definitions we can now write down (3.2) after performing the baryon number and isospin reduction

$$\left( i \gamma^\mu \frac{\partial}{\partial x^\mu} \right) \tau(x, y) \left( i \gamma^\nu \frac{\partial}{\partial y^\nu} \right) \\ = [6 - T(T+1)] l^4 F_\mu(x - y) F_\nu(x - y) \frac{1}{2} [\gamma^\mu \tau(x, y) \gamma^\nu + \gamma_5 \gamma^\mu \tau(x, y) \gamma_5 \gamma^\nu] \\ - \frac{1}{2} [6 - T(T+1)] l^4 F_\mu(x - y) F^\mu(x - y) \frac{1}{2} [\gamma^\lambda \tau(x, y) \gamma_\lambda + \gamma_5 \gamma^\lambda \tau(x, y) \gamma_5 \gamma_\lambda] \\ + [6 - 2T(T+1)] l^4 F_\mu(x - y) F^\mu(x - y) \frac{1}{2} [C \tau^T(y, x) C + \gamma_5 C \tau^T(y, x) \gamma_5 C]. \quad (3.8)$$

We have included a factor  $\beta$  in the definition of  $\tau$  so that it corresponds to the usual form of a  $\tau$ -function:

$$\tau(x, y) := \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle. \quad (3.9)$$

In (3.8) we can now identify the different terms: The first two interaction terms are direct ones, the last one is an exchange term. The spin structure of the individual terms is that of a usual  $V-A$ -theory. This could have been derived also from (2.4) where it is obscured by the algebra.

The Dirac operators on the left-hand side of (3.8) contain no mass term corresponding to the vanishing bare mass of the nucleon in (2.1). This fact will enable us to reduce (3.8) to the Weyl-form. For the moment, we neglect the exchange term which will not change the argument. Furthermore, we introduce:

$$D_{\mu\nu}(x-y) = [6 - T(T+1)] l^4 F_\mu(x-y) F_\nu(x-y) - \frac{1}{2} [6 - T(T+1)] l^4 F_\lambda(x-y) F^\lambda(x-y) g_{\mu\nu}. \quad (3.10)$$

(3.8) reads then:

$$\left( i \gamma^\mu \frac{\partial}{\partial x^\mu} \right) \tau(x, y) \left( i \gamma^\nu \frac{\overleftarrow{\partial}}{\partial y^\nu} \right) = D_{\mu\nu}(x-y) \gamma^\mu \tau(x, y) \gamma^\nu + D_{\mu\nu}(x-y) \gamma_5 \gamma^\mu \tau(x, y) \gamma_5 \gamma^\nu. \quad (3.11)$$

If we work in the representation of the  $\gamma$ -matrices with  $\gamma_5$  diagonal, we can decompose  $\psi(x)$  into left- and right-handed part by:

$$\psi(x) = \frac{1}{2}(1 + \gamma_5)\psi(x) + \frac{1}{2}(1 - \gamma_5)\psi(x) = \xi(x) + \eta(x) = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix}. \quad (3.12)$$

For the  $\tau$ -function (3.9), this implies the decomposition:

$$\langle 0 | T \psi(x) \bar{\psi}(y) | a \rangle = \langle 0 | T \begin{pmatrix} \xi(x) \eta^+(y) & \xi(x) \xi^+(y) \\ \eta(x) \eta^+(y) & \eta(x) \xi^+(y) \end{pmatrix} | a \rangle = : \tau_{ik}(x, y). \quad (3.13)$$

From (3.11) we can now derive a set of equations for the  $\tau_{ik}$  which will decouple because of chiral invariance. They read:

$$\begin{aligned} \left( i \sigma^\mu \frac{\partial}{\partial x^\mu} \right) \tau_{11}(x, y) \left( i \bar{\sigma}^\nu \frac{\overleftarrow{\partial}}{\partial y^\nu} \right) &= 0, \\ \left( i \sigma^\mu \frac{\partial}{\partial x^\mu} \right) \tau_{12}(x, y) \left( i \sigma^\nu \frac{\overleftarrow{\partial}}{\partial y^\nu} \right) &= 2 D_{\mu\nu}(x-y) \sigma^\mu \tau_{12}(x, y) \sigma^\nu, \\ \left( i \bar{\sigma}^\mu \frac{\partial}{\partial x^\mu} \right) \tau_{21}(x, y) \left( i \bar{\sigma}^\nu \frac{\overleftarrow{\partial}}{\partial y^\nu} \right) &= 2 D_{\mu\nu}(x-y) \bar{\sigma}^\mu \tau_{21}(x, y) \bar{\sigma}^\nu, \\ \left( i \bar{\sigma}^\mu \frac{\partial}{\partial x^\mu} \right) \tau_{22}(x, y) \left( i \sigma^\nu \frac{\overleftarrow{\partial}}{\partial y^\nu} \right) &= 0, \end{aligned} \quad (3.14)$$

Hence we get free equations for  $\tau_{11}$  and  $\tau_{22}$ . Since they do not possess normalizable solutions we conclude  $\tau_{11} = \tau_{22} \equiv 0$ . The equations for  $\tau_{12}$  and  $\tau_{21}$  yield identical eigenvalues since they differ only in the sign of  $\sqrt{s}$  after expansion in terms of a  $\bar{\sigma}$ - and  $\sigma$ -algebra, respectively. This sign is without relevance (see also Section 5). So it will suffice to consider the equation for  $\tau_{12}$ .

Including the exchange term again, we have to solve therefore the following equation, dropping the indices on  $\tau_{12}$ :

$$\begin{aligned} \left( i \sigma^\mu \frac{\partial}{\partial x^\mu} \right) \tau(x, y) \left( i \sigma^\nu \frac{\overleftarrow{\partial}}{\partial y^\nu} \right) &= [6 - T(T+1)] l^4 F_\mu(x-y) F_\nu(x-y) \sigma^\mu \tau(x, y) \sigma^\nu \\ &\quad - \frac{1}{2} [6 - T(T+1)] l^4 F_\mu(x-y) F^\mu(x-y) \sigma^\lambda \tau(x, y) \sigma_\lambda \\ &\quad + [6 - 2T(T+1)] l^4 F_\mu(x-y) F^\mu(x-y) c_\sigma \tau^T(y, x) c_\sigma. \end{aligned} \quad (3.15)$$

We would have obtained this equation directly by starting with the Weyl form of (2.1). This reduction is of great practical importance since it reduces the number of equations to be considered by a factor 4.

The next step in solving (3.15) is the separation of the center of mass motion by the ansatz

$$\tau(x, y) = \exp \{ -i I(x+y)/2 \} \psi(x-y). \quad (3.16)$$

This yields the equation for the relative coordinate wave function  $\psi(z)$ :

$$\begin{aligned} \left( i \sigma^\mu \frac{\partial}{\partial z^\mu} + \frac{1}{2} \sigma^\mu I_\mu \right) \psi(z) \left( -i \sigma^\nu \frac{\overleftarrow{\partial}}{\partial z^\nu} - \frac{1}{2} \sigma^\nu I_\nu \right) &= [6 - T(T+1)] l^4 F_\mu(z) F_\nu(z) \sigma^\mu \psi(z) \sigma^\nu \\ &\quad - \frac{1}{2} [6 - T(T+1)] l^4 F_\mu(z) F^\mu(z) \sigma^\lambda \psi(z) \sigma_\lambda + [6 - 2T(T+1)] l^4 F_\mu(z) F^\mu(z) c_\sigma \psi^T(-z) c_\sigma. \end{aligned} \quad (3.17)$$

In this equation, one has now to perform the angular momentum reduction, as described in <sup>9</sup>. For the numerical solution intended these methods are not very convenient since a two-variable partial differential equation results. Therefore, we will perform in the next section the Gourdin expansion immediately in (3.17) which of course contains an angular momentum reduction, too.

#### 4. The Gourdin Expansion

All known numerical methods for solving the Bethe-Salpeter equation make use of the Wick rotation <sup>26</sup>, i. e. the equation is transformed to the four-dimensional Euclidian space. The reason for this depends on the actual computational method: in the momentum space integral equation it removes the singularities of the kernel thus rendering the kernel square-integrable. On the contrary, the position space differential equation is transformed from a hyperbolic equation to an elliptic one which allows imposing boundary conditions at the origin and at infinity.

We will also perform the Wick rotation. Some problems related to the question of its validity are discussed in Appendix I. Here we will take it for granted. So we introduce the Wick-rotated coordinates

$$x_1 = z^1, \quad x_2 = z^2, \quad x_3 = z^3, \quad x_4 = i z^0; \quad R^2 = x_\mu x_\mu \quad (4.1)$$

and the transformed Pauli matrices

$$\tilde{\sigma}_1 = -i \sigma^1, \quad \tilde{\sigma}_2 = -i \sigma^2, \quad \tilde{\sigma}_3 = -i \sigma^3, \quad \tilde{\sigma}_4 = \sigma^0 \quad (4.2)$$

fulfilling the commutation relation

$$\tilde{\sigma}_\mu \tilde{\sigma}_\nu + \sigma_\nu \tilde{\sigma}_\mu = 2 \delta_{\mu\nu}. \quad (4.3)$$

The transformed wave function will be denoted by  $\varphi(x)$ . The Bethe-Salpeter equation reads now:

$$\begin{aligned} & (-\sigma_\mu \partial_\mu + \frac{1}{2} \tilde{\sigma}_\mu I_\mu) \varphi(x) (-\sigma_\nu \tilde{\partial}_\nu + \frac{1}{2} \sigma_\nu I_\nu) \\ & = -g^4 [3 - \frac{1}{2} T(T+1)] V^2(R) (\delta_{\mu\nu} + \hat{x}_\mu \hat{x}_\nu) \tilde{\sigma}_\mu \varphi(x) \tilde{\sigma}_\nu - g^4 [3 - T(T+1)] V^2(R) c_\sigma \varphi^T(-x) c_\sigma. \end{aligned} \quad (4.4)$$

$I_\mu = (0, 0, 0, E)$  is the (non-Wick-rotated) center of mass momentum in the rest frame,  $\hat{x}_\mu$  the unit vector corresponding to  $x_\mu$ . The coupling constant is

$$g = \kappa l / 2 \pi. \quad (4.5)$$

$V(R)$  plays the role of a relativistic potential. It is related to  $F_\mu(x)$  by

$$(2\pi)^2 F_\mu(x) = \hat{x}_\mu \kappa^2 V(R) \quad (4.6)$$

and from standard integrals it follows:

$$V(R) = -(1/R) K_2(\kappa R) + 2/(\kappa^2 R^3) - 1/(2R). \quad (4.7)$$

$K_2$  denotes the modified Bessel function.  $V(R)$  is shown in Figure 1. It is drawn at an attractive potential though this is probably not an important distinction for spinorial Bethe-Salpeter equations: They give rise to bound states for both signs of the interaction <sup>13</sup>.

The wave function is now expanded in terms of a  $\tilde{\sigma}$ -algebra:

$$\varphi(x) = \tilde{\sigma}^\mu \varphi_\mu(x). \quad (4.8)$$

With the abbreviations

$$V^{(D)}(R) = 4 g^4 [3 - T(T+1)] V^2(R), \quad (4.9)$$

$$V^{(E)}(R) = 2 g^4 [3 - \frac{1}{2} T(T+1)] V^2(R), \quad (4.10)$$

we have after the algebraic evaluation

$$\begin{aligned} & \left( \square - \frac{E^2}{4} \right) \varphi_\mu(x) - 2 \partial_\mu \partial_\nu \varphi_\nu(x) + \frac{1}{2} I_\mu I_\nu \varphi_\nu(x) - \varepsilon_{\mu\nu\sigma} I_\nu \partial_\sigma \varphi_\sigma(x) \\ & = V^{(D)}(R) \varphi_\mu(x) - V^{(D)}(R) \hat{x}_\mu \hat{x}_\nu \varphi_\nu(x) + V^{(E)}(R) \varphi_\mu(-x). \end{aligned} \quad (4.11)$$

In the rest frame this is separated into the scalar ( $\mu=4$ ) and the vector ( $\mu=1, 2, 3$ ) part:

$$\left( \square - 2 \partial_4^2 + \frac{E^2}{4} + V^{(D)}(R) - V^{(D)}(R) x_4 \hat{x}_4 \right) \varphi_4(x) - V^{(E)}(R) \varphi_4(-x) + [-2 \partial_4 \partial_j + V^{(D)}(R) \hat{x}_4 \hat{x}_j] \varphi_j(x) = 0, \quad (4.12)$$

$$\left( \square - \frac{E^2}{4} - V^{(D)}(R) \right) \varphi_j(x) - V^{(E)}(R) \varphi_j(-x) + [-2 \partial_j \partial_k + V^{(D)}(R) x_j \hat{x}_k] \varphi_k(x) + \varepsilon_{jkl} E \partial_k \varphi_l(x) + [-2 \partial_j \partial_4 + V^{(D)}(R) \hat{x}_j \hat{x}_4] \varphi_4(x) = 0. \quad (4.13)$$

As the next step, we perform now the Gourdin expansion<sup>27</sup>, i. e. we expand  $\varphi_4(x)$  and  $\varphi_j(x)$  in terms of four-dimensional scalar and vector spherical harmonics. There are other methods employed in the literature for the  $O(4)$ -expansion (see e. g.<sup>28</sup>) but the Gourdin expansion is most suited for numerical calculations.

The four-dimensional scalar spherical harmonics are defined by:

$$Y_{Njj_3}(\Theta, \vartheta, \varphi) = G_N^{(j)}(\Theta) Y_{jj_3}(\vartheta, \varphi) \quad (4.14)$$

$$\text{with} \quad G_N^{(j)}(\Theta) = \sqrt{\frac{2^{2j+1}(N+1)(N-j)!}{\pi(N+j+1)!}} j! \sin^j \Theta C_{N-j}^{(j+1)}(\cos \Theta). \quad (4.15)$$

$C_n^{(v)}(\cos \Theta)$  are the usual Gegenbauer polynomials<sup>29</sup>, and  $Y_{jj_3}(\vartheta, \varphi)$  the three-dimensional spherical harmonics. The angle  $\Theta$  is given by  $x_4 = R \cos \Theta$ . The four-dimensional vector spherical harmonics are defined by:

$$Y_{(N)ljj_3 i}(\Theta, \vartheta, \varphi) = G_N^{(l)}(\Theta) Y_{ljj_3 i}(\vartheta, \varphi). \quad (4.16)$$

The  $Y_{(l)jj_3 i}(\vartheta, \varphi)$  are the well-known three-dimensional vector spherical harmonics<sup>30</sup>.

According to Gourdin, we now make the following ansatz for the wave-function:

$$\varphi_4(x) = \sum_{N=j}^{\infty} \varphi_{N-jjj_3}^{(4)}(R) Y_{Njj_3}(\Theta, \vartheta, \varphi), \quad (4.17)$$

$$\varphi_k(x) = \sum_{N=j}^{\infty} \{ \varphi_{N-jjj_3}^{(+)}(R) Y_{(N+1)jj_3 k}(\Theta, \vartheta, \varphi) + i \varphi_{N-jjj_3}^{(0)}(R) Y_{(N)jj_3 k}(\Theta, \vartheta, \varphi) + \varphi_{N-jjj_3}^{(-)}(R) Y_{(N-1)jj_3 k}(\Theta, \vartheta, \varphi) \}. \quad (4.18)$$

This ansatz is now inserted into (4.12) (4.13). This will yield different operators acting on the four-dimensional spherical harmonics like differentiation and multiplication by unit vectors. Because of the completeness of the functions (4.14) (4.16), the results of these operations can all be expressed as linear combinations of functions with different index  $N$  ( $j$  and  $j_3$  are not changed because of rotational invariance). All relevant formulas are collected in<sup>16</sup>. The only exception are the exchange terms. For them we need the behaviour of the spherical harmonics under total reflections  $x \rightarrow -x$ :

$$Y_{Njj_3}(\pi - \Theta, \pi - \vartheta, \pi + \varphi) = (-1)^N Y_{Njj_3}(\Theta, \vartheta, \varphi), \quad (4.19)$$

$$Y_{(N)ljj_3 k}(\pi - \Theta, \pi - \vartheta, \pi + \varphi) = (-1)^N Y_{(N)ljj_3 k}(\Theta, \vartheta, \varphi). \quad (4.20)$$

The equations for  $\varphi_{N-jjj_3}^{(i)}(R)$  are then derived with help of the orthonormality relations for the spherical harmonics. In the following we will suppress the indices  $j$  and  $j_3$  and set  $n = N - j$ , which is always an integer even for not integer  $j$ .

As result of these manipulations we obtain the following equations for the  $\varphi_n^{(i)}(R)$ :

$$\left\{ d(0; n+j, R) + \frac{E^2}{4} - V^{(D)}(R) - (-1)^{n+j} V^{(E)}(R) \right\} \varphi_n^{(4)}(R) + \sum_{k=-1}^1 Q(2k, 0; n+j-2k, j) \{ -2d(-2k; n+j-2k, R) + V^{(D)}(R) \} \varphi_{n-2k}^{(4)}(R) + \sum_{k=-1}^1 Q(-2k, 1; n+j, j) \{ -2d(-2k; n+j-2k, R) + V^{(D)}(R) \} \varphi_{n-2k-1}^{(+)}(R) + \sum_{k=-1}^1 Q(-2k, -1; n+j, j) \{ -2d(-2k; n+j-2k, R) + V^{(D)}(R) \} \varphi_{n-2k+1}^{(-)}(R) = 0; \quad (4.21)$$



$$\left\{ d(0; n+j, R) - \frac{E^2}{4} - V^{(D)}(R) - (-1)^{n+j} V^{(E)}(R) \right\} \varphi_n^{(0)}(R) - E \{ X(-1, 1; n+j, j) d(-1; n+j-1, R) \varphi_{n-2}^{(+)}(R) + X(1, 1; n+j, j) d(1; n+j+1, R) \varphi_n^{(+)}(R) \} - E \{ X(-1, -1; n+j, j) d(-1; n+j-1, R) \varphi_n^{(-)} + X(1, -1; n+j, j) d(1; n+j+1, R) \varphi_{n+2}^{(-)}(R) \} = 0; \quad (4.22)$$

$$\left\{ d(0; n+j+1, R) - \frac{E^2}{4} - V^{(D)}(R) - (-1)^{n+j} V^{(E)}(R) \right\} \varphi_n^{(+)}(R) + E \{ X(1, 1; n+j, j) d(-1; n+j, R) \varphi_n^{(0)}(R) + X(-1, 1; n+j+2, j) d(1; n+j+2, R) \varphi_{n+2}^{(0)}(R) \} + \sum_{k=-1}^1 W^{(+)}(2k, 1; n+j-2k+1, j) \{ -2d(-2k; n+j-2k+1, R) + V^{(D)}(R) \} \varphi_{n-2k}^{(+)}(R) + \sum_{k=-1}^1 W^{(-)}(2k, 1; n+j-2k+1, j) \{ -2d(-2k; n+j-2k+1, R) + V^{(D)}(R) \} \varphi_{n-2k+2}^{(-)}(R) + \sum_{k=-1}^1 Q(2k, 1; n+j-2k+1, j) \{ -2d(-2k; n+j-2k+1, R) + V^{(D)}(R) \} \varphi_{n-2k+1}^{(4)}(R) = 0; \quad (4.23)$$

$$\left\{ d(0; n+j-1, R) - \frac{E^2}{4} - V^{(D)}(R) - (-1)^{n+j} V^{(E)}(R) \right\} \varphi_n^{(-)}(R) + E \{ X(1, -1; n+j-2, j) d(-1; n+j-2, R) \varphi_{n-2}^{(0)}(R) + X(-1, -1; n+j, j) d(1; n+j, R) \varphi_n^{(0)}(R) \} + \sum_{k=-1}^1 W^{(+)}(2k, -1; n+j-2k-1, j) \{ -2d(-2k; n+j-2k-1, R) + V^{(D)}(R) \} \varphi_{n-2k-2}^{(+)}(R) + \sum_{k=-1}^1 W^{(-)}(2k, -1; n+j-2k-1, j) \{ -2d(-2k; n+j-2k-1, R) + V^{(D)}(R) \} \varphi_{n-2k}^{(-)}(R) + \sum_{k=-1}^1 Q(2k, -1; n+j-2k-1, j) \{ -2d(-2k; n+j-2k-1, R) + V^{(D)}(R) \} \varphi_{n-2k-1}^{(4)}(R) = 0. \quad (4.24)$$

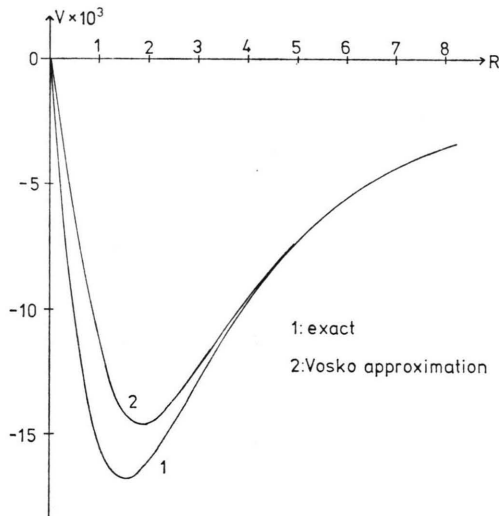


Fig. 1. The potential  $V(R)$  (4.7) together with the Vosko approximation (5.14).

The coefficients  $Q$ ,  $X$ , and  $W^{(\pm)}$  are kinematical factors (essentially four-dimensional Clebsch-Gordan coefficients) which are defined in <sup>16</sup>. The  $d(k; n, R)$  are differential operators of first ( $k = \pm 1$ ) and second order ( $k = 0, \pm 2$ ) which are defined in <sup>16</sup>, too.

For  $j=0$  the dynamical equations will reduce to two equations for  $\varphi_n^{(4)}(R)$  and  $\varphi_n^{(+)}(R)$  since (4.21) (4.23) will decouple from the two other equations because of the vanishing of the kinematical coefficients. In fact, for  $j=0$  we have  $\varphi_n^{(0)}(R) = \varphi_n^{(-)}(R) = 0$  from the definition of the vector spherical harmonics.

With (4.21) – (4.24) the equations are in a form which is suitable for numerical computations. They constitute an (infinite) system of coupled ordinary differential equations which can be solved by a variational procedure.

We will close this section with some remarks on the quantum number assignment to the eigenvalues. The angular momentum of them is given by the parameter  $j$ . More intricate is the question of discrete symmetries. Since according to Section 3 the solutions are parity degenerate there remains only the  $CP$ -invariance. It is shown in <sup>12</sup> that the  $CP$ -eigenvalue  $\eta_{CP}$  is related to the index  $n$  of the  $\varphi_n^{(i)}(R)$  in the following manner:

$$\eta_{CP} = +1: \varphi_{2n+1}^{(4)}(R), \varphi_{2n}^{(+)}(R), \varphi_{2n}^{(0)}(R), \varphi_{2n}^{(-)}(R), \quad (4.25)$$

$$\eta_{\text{CP}} = -1: \varphi_{2n}^{(4)}(R), \varphi_{2n+1}^{(+)}(R), \varphi_{2n+1}^{(0)}(R), \varphi_{2n+1}^{(-)}(R). \quad (4.26)$$

A careful examination of (4.21) – (4.24) shows that it splits up indeed in the indicated fashion.

In the following we will fix the mass scale by requiring  $\kappa=1$  and introduce the variable  $s=E^2/\kappa^2$ .

## 5. Variational Method

Since we intend to solve (4.21) – (4.24) by a variational procedure we have to choose a set of suitable trial functions. It is essential that they match the true solution as close as possible so that a small set of functions will yield acceptable results. A suitable set may be found by studying the asymptotic behaviour of the solution for  $R \rightarrow 0$  and  $R \rightarrow \infty$ . Such an investigation has been performed in<sup>17</sup>, and we will not repeat it here. By these considerations, the following set of trial functions is suggested:

$$\varphi_n^{(4)}(R) = \sum_{h=0}^{\infty} \tilde{\varphi}_{nh}^{(4)} R^{n+j+h} e^{-\alpha R}, \quad (5.1)$$

$$\varphi_n^{(+)}(R) = \sum_{h=0}^{\infty} \tilde{\varphi}_{nh}^{(+)} R^{n+j+h+1} e^{-\alpha R}, \quad (5.2)$$

$$\varphi_n^{(0)}(R) = \sum_{h=0}^{\infty} \tilde{\varphi}_{nh}^{(0)} R^{n+j+h} e^{-\alpha R}, \quad (5.3)$$

$$\varphi_n^{(-)}(R) = \sum_{h=0}^{\infty} \tilde{\varphi}_{nh}^{(-)} R^{n+j+h-1} e^{-\alpha R}. \quad (5.4)$$

For the numerical solution the sum over  $h$  [and also the sum over  $n$  in (4.17) (4.18)] will be truncated at a finite value.

The system (4.21) – (4.24) can be written in matrix notation in an obvious way:

$$\sum_{i=1}^4 \sum_n \hat{B}(i', n' | i, n) \varphi_n^{(i)}(R) = 0. \quad (5.5)$$

In (5.5) we now insert the ansatz (5.1) – (5.4). Then we multiply it from the left with  $\varphi_n^{(i')}(R)$  [again in the form (5.1) – (5.4)] and integrate with respect to  $R$ . This yields a system of algebraic equations for the  $\tilde{\varphi}_{nh}^{(i)}$ :

$$\sum_{i=1}^4 \sum_{i'=1}^4 \sum_n \sum_{n'} \sum_h \sum_{h'} \tilde{\varphi}_{n'h'}^{(i')} \cdot \tilde{B}(i', n', h' | i, n, h) \tilde{\varphi}_{nh}^{(i)} = 0. \quad (5.6)$$

The matrix elements of  $\tilde{B}$  are read off trivially from (4.21) – (4.24). A sufficient condition for

(5.6) to be fulfilled is:

$$\sum_{i=1}^4 \sum_n \sum_h \tilde{B}(i', n', h' | i, n, h) \tilde{\varphi}_{nh}^{(i)} = 0. \quad (5.7)$$

We can not conclude that (5.7) is also necessary for the validity of (5.6), since we do not know whether  $\tilde{B}$  is positive definite.

In order to construct an usual eigenvalue problem out of (5.7) we follow the approach suggested in<sup>31</sup>. (5.7) may be considered as a nonlinear eigenvalue problem in  $\sqrt{s}$  which is in an abbreviated notation:

$$(\tilde{P}_0 + \sqrt{s} \tilde{P}_1 + s \tilde{P}_2) = 0. \quad (5.8)$$

This can be transformed into a linear problem by doubling the number of components: We define:

$$\tilde{\varphi}_1 := \varphi, \tilde{\varphi}_2 := \sqrt{s} \varphi, \tilde{\Phi} := \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix} \quad (5.9)$$

and instead of (5.8) we get the system

$$\left[ \begin{pmatrix} \tilde{P}_0 & 0 \\ 0 & \tilde{P}_2 \end{pmatrix} + \sqrt{s} \begin{pmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_2 & 0 \end{pmatrix} \right] \tilde{\Phi} = 0. \quad (5.10)$$

(5.10) has the disadvantage that it is an eigenvalue equation for  $\sqrt{s}$  and not for the physically meaningful Mandelstam variable  $s$ . This can easily be remedied by iterating (5.10). So we obtain finally the eigenvalue equation:

$$\begin{pmatrix} -\tilde{P}_2^{-1} \tilde{P}_0 & -\tilde{P}_2^{-1} \tilde{P}_1 \\ \tilde{P}_2^{-1} \tilde{P}_1 \tilde{P}_2^{-1} \tilde{P}_0 & -\tilde{P}_2^{-1} \tilde{P}_0 + \tilde{P}_2^{-1} \tilde{P}_1 \tilde{P}_2^{-1} \tilde{P}_1 \end{pmatrix} \tilde{\Phi} = s \tilde{\Phi}. \quad (5.11)$$

This is the equation from which we calculated the eigenvalues.

The calculation of the matrix elements is trivial concerning the kinetic part. All matrix elements lead to integrals of the following type:

$$\int_0^{\infty} dR R^{\beta} e^{-\alpha R} = \Gamma(\beta+1) / \alpha^{\beta+1}. \quad (5.12)$$

In the calculation of the interaction term matrix elements we encounter integrals of the type

$$\int_0^{\infty} dR R^{\beta} e^{-\alpha R} V^2(R) \quad (5.13)$$

with  $V(R)$  given by (4.7). In order to avoid an unstable numerical integration we introduced an approximation for  $V(R)$  following<sup>32</sup> so that we can calculate (5.13) analytically:

$$V(R) = - \left( \frac{1}{R^3} + \frac{3}{R^2} \right) e^{-R} - \frac{1}{R^3} e^{-2R} + \frac{2}{R^3} - \frac{1}{2R}. \quad (5.14)$$

This "Vosko-approximation" is also shown in Figure 1. It deviates from the exact potential only for small values of  $R$ . The error in (5.13) was found to be negligible. For more details on this see <sup>33</sup>.

The next problem is that of the truncation of the infinite sums over  $n$  and  $h$ . To that end we calculate the eigenvalues for different upper limits  $N$  and  $H$ . Then  $N$  and  $H$  are raised until some of the eigenvalues do not change anymore. These eigenvalues are regarded as physical whereas those changing rapidly from order to order are considered as a mere effect of the approximation. A numerical example for the convergence is given in the next section. We have found stability of the eigenvalues for 3 or 4 terms in the sum over  $n$ . For the sum over  $h$ , we have tested two possibilities. Firstly we have chosen  $H$  to be independent of  $n$ , and secondly we have set  $H = N' - n$  with  $N' \geq N$ . So in the second method we have more trial functions for low  $0(4)$  quantum numbers. This method turned out to be much better yielding stable results for smaller matrices. For  $n=0$  we used 5 or 6 trial function and correspondingly less for higher values of  $n$ .

The nonlinear parameter  $\alpha$  was fixed in such a way that a fast convergence results. As a convenient value we found  $\alpha = 1$ . All results of the next sections were obtained with this  $\alpha$ .

## 6. Results for Scalar Mesons

For scalar mesons ( $j=0$ ) the Eq. (4.21) – (4.24) are simplified considerably since in this case we have only two non-vanishing wave-functions [ $\varphi^{(4)}(R)$  and  $\varphi^{(+)}(R)$ ]. Therefore (4.21) (4.23) decouple from the two other equations. So we have  $\tilde{P}_1 \equiv 0$  and instead of (5.11) we find:

$$\begin{pmatrix} -\tilde{P}_2^{-1}\tilde{P}_0 & 0 \\ 0 & -\tilde{P}_2^{-1}\tilde{P}_0 \end{pmatrix} \tilde{\Phi} = s \tilde{\Phi}. \quad (6.1)$$

So the eigenvalue problem is reduced in a trivial fashion. For  $j=0$ , the matrices are therefore smaller by a factor 4 compared to the general case.

The CP-eigenvalue is fixed to  $\eta_{CP} = -1$ . As is explained in <sup>10</sup>, for this CP-eigenvalue also the first order contact graph will contribute. So it will be possible to investigate the validity of the Neumann expansion (2.16).

But for the moment we still concentrate on the second order. For the  $\eta$ -meson ( $T=0$ ) we have performed extensive numerical tests concerning the stability of the results with respect to the variation of the parameter  $\alpha$ ,  $N$ , and  $H$  and concerning the dependence on  $g^4$ .

The number of eigenvalues depends of course on the order of matrices. The number of real eigenvalues, however, is only sensitive to the magnitude of the coupling constant  $g^4$ . All complex eigenvalues change rapidly with increasing matrix size whereas the real eigenvalues tend to stabilize. Therefore we consider the complex eigenvalues as unphysical.

For  $g^4 < 1.7$  we did not find real eigenvalues at all, beyond these values we got two of them. For very high  $g^4$  values ( $g^4 > 10$ ) we found even four or more which were partly unstable. Since we have always used an even matrix size we will automatically get an even number of real eigenvalues. It should not be considered as a defect that we got no eigenvalues for the physical range of  $g^4$  ( $g^4 \sim 1$ ) since only the combination of first and second order is meaningful.

For  $g^4 = 5$ , the effect of varying the matrix size on the eigenvalues is shown in Table 1.  $N$  is the number of  $0(4)$  eigenfunctions [the sum over  $n$  runs in steps of 2 because of CP-invariance (4.25) (4.26)].  $H$  is the number of trial functions for the highest  $n$  value. It increases by 1 for each lower  $n$ . Finally, we have chosen  $\alpha = 1$ .

Table 1. Dependence of the ground state and first excited state masses on the matrix size. The parameters are described in the text.

$N$	$H$	Matrix Size	Eigenvalues	
1	5	12×12	0.195744	5.16167
2	4	22×22	0.224529	7.02591
3	3	30×30	0.256193	6.80726
4	2	36×36	0.256575	7.05457
5	1	40×40	0.256577	7.35293

For the ground state, we get a 25% accuracy already for  $N=1$  and for  $N=3$  an almost correct result. The convergence of the eigenvalues is thus highly satisfactory and comparable to the scalar case <sup>15</sup>. The results for the first excited state are not so precise as to be expected.

For the  $\pi$ - and  $\eta$ -meson the results are shown in Figure 2. Only the lowest eigenvalue has been drawn. Hence we got real eigenvalues also above threshold at  $s=0$  as already found in <sup>31</sup>. We found

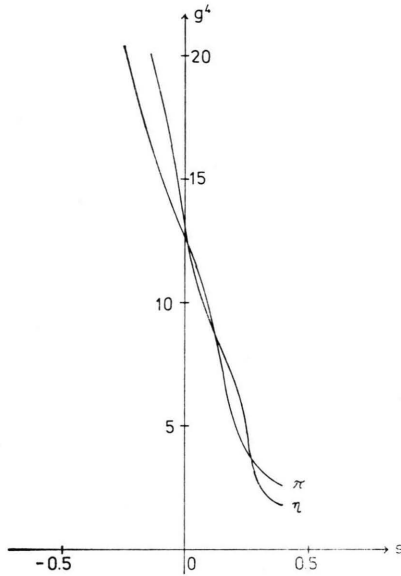


Fig. 2.  $g^4$  dependent on  $s$  for the  $\pi$ - and  $\eta$ -meson by taking into account only the second order.

no difference in stability for  $s \geq 0$ , so we have no reason to reject these solutions.

Now we take into account also the local graph. The result for it, taken apart, can be written for the  $\eta$ -meson:

$$1 + \frac{3}{4} g^2 q_1(s) = 0 \quad (6.2)$$

with the function

$$q_1(s) = \frac{1}{2} \left\{ -\frac{1}{s} + \ln|s| - \frac{(1-s)^2}{s^2} \ln|1-s| \right\}. \quad (6.3)$$

Compared to usual first order calculations<sup>1, 23</sup> we have an additional factor  $\frac{1}{2}$  which arises from the different multiplicities of the contact graph in (2.17) and (2.18).

Similarly we can write the result for the nonlocal graph in the form:

$$1 - \frac{9}{16} g^4 q_2(s) = 0. \quad (6.4)$$

The function  $q_2(s)$  is taken from Figure 2. If we combine both terms we get:

$$1 + \frac{3}{4} g^2 q_1(s) - \frac{9}{16} g^4 q_2(s) = 0. \quad (6.5)$$

This can be solved for  $g^2$ :

$$g^2 = \frac{2}{3} \frac{q_1(s)}{q_2^2(s)} - \sqrt{\frac{4}{9} \frac{q_1^2(s)}{q_2^4(s)} + \frac{16}{9} \frac{1}{q_2^2(s)}}. \quad (6.6)$$

The sign of the square root is chosen such that we get back (6.2) for  $q_2(s) \rightarrow 0$ . If we define a function  $q_{12}(s)$  by:

$$\frac{1}{q_{12}(s)} = -\frac{1}{2} \frac{q_1(s)}{q_2^2(s)} \left( 1 - \sqrt{1 + 4 \frac{q_2^2(s)}{q_1^2(s)}} \right) \quad (6.7)$$

we can write (6.5) in the form

$$1 + \frac{3}{4} g^2 q_{12}(s) = 0. \quad (6.8)$$

The functions  $q_1(s)$ ,  $q_2(s)$ , and  $q_{12}(s)$  are shown in Fig. 3 for the  $\eta$ -meson and in Fig. 4 for the  $\pi$ -meson. The effect of the nonlocal graph can now be studied by comparing  $q_1(s)$  and  $q_{12}(s)$ .

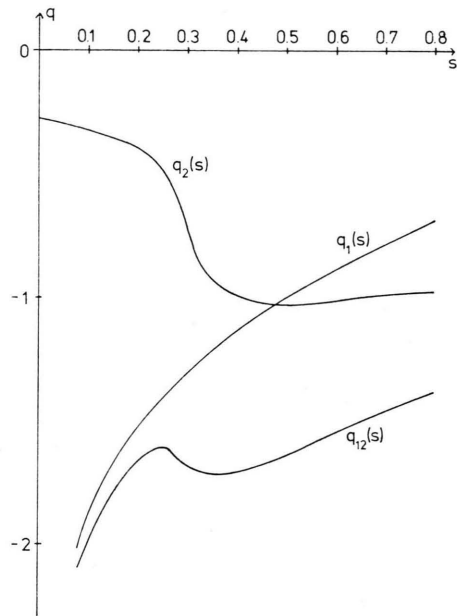
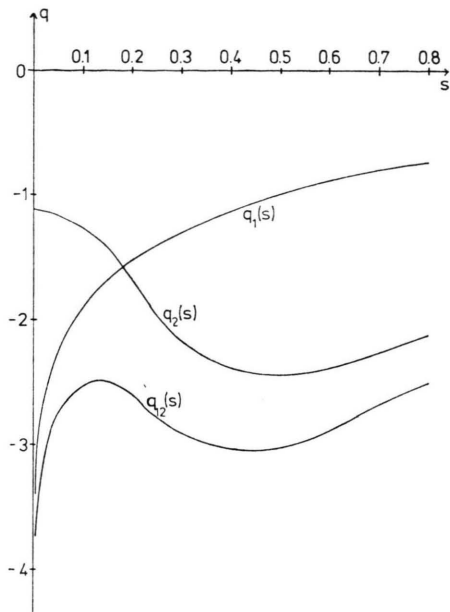


Fig. 3. The functions  $q_1(s)$ ,  $q_2(s)$ , and  $q_{12}(s)$  [for their definition see (6.3), (6.4), and (6.7)] for the  $\eta$ -meson.

The difference between  $q_1(s)$  and  $q_{12}(s)$  is smallest for small values of  $s$ . The functions themselves are largest in this region, hence the coupling constant is small according to (6.8). So the effect of the nonlocal graph is smallest for small coupling constants, a very satisfactory result. For larger values of  $s$  and hence also of  $g$  the nonlocal graph cannot be neglected for quantitative statements, but it is still a correction to the local one and does not change the results by orders of magnitude as it was found for the Fredholm approximation.

By the nonlocal graph the meson masses are shifted to higher  $s$  values. It is difficult to make



Fig. 4. The same as Fig. 3 for the  $\pi$ -meson.

definite statements about their magnitude. For this we need the coupling constant from a second order calculation. A complete calculation of this kind does not exist, only a first attempt has been made<sup>34</sup>. There it was found

$$g^2 = 0.845 \quad (6.9)$$

whereas in first order it is<sup>1</sup>

$$g^2 = 1.034 \quad (6.10)$$

In spite of its uncertainty we will use (6.9) to make definite predictions. The resulting mass values are listed in Table 2 together with the first order result. The agreement with the experimental masses is much better than for the first order.

Table 2. The masses for the  $j=0$ ,  $\eta_{CP}=-1$  mesons, calculated from the first order ( $m_1$ ) and from the combined first and second order ( $m_{12}$ ).

$T$	Particle	$m_1$ [MeV]	$m_{12}$ [MeV]	exp. [MeV]
0	$\eta$	865	470	549
0	$\eta'$	—	734	958
1	$\pi$	291	38	140

In the isoscalar channel we find in addition an excited state which may be identified with the  $\eta'$ . The corresponding masses are also given in Table 2. This mass, too, has the correct order of magnitude. No excited state is found in the isovector case which agrees with the experimental situation.

Concluding this section, we may say that we have confirmed the existence of scalar mesons in non-linear spinor theory and have demonstrated the validity of the Neumann expansion. The main effect of the second order is the change in the factor of the local graph and not the addition of the nonlocal graphs.

## 7. Calculation of Regge Trajectories

In the second order we get not only scalar mesons, but also particles with an arbitrary angular momentum. The Eqs. (4.21) – (4.24) and the computer program are designed such that one is able to handle the general case. As an example for this, we present in this section a calculation of a Regge trajectory. Before we present the result, we will comment briefly on the problems connected with the analytic continuation in  $j$ .

The factor  $(-1)^j$  in (4.21) – (4.24) arises from the existence of an exchange term in (3.17). Hence we have to introduce the signature  $\tau$  of the trajectory. For a right signature point we have  $\tau = (-1)^j$ , whereas for a wrong signature point it is  $\tau = -(-1)^j$ . The signature can be taken into account by replacing  $(-1)^j$  by  $\tau$  in (4.21) – (4.24) and treating  $\tau$  as an additional parameter.

Table 3. The different possible Regge trajectories and their identification with physical particles.

$\eta_{CP}$	$\tau$	Particles ( $T=0$ )	Particles ( $T=1$ )
+1	+1	$f$	$A_2$
+1	-1	$\omega$	$\varrho$
-1	+1	$\eta$	$\pi$
-1	-1	unknown	$B$

All possible trajectories are listed in Table 3. The identification with particles is only tentative because of the unphysical parity degeneracy. Since the isoscalar particles are not quite well-known we concentrate on the isovector case.

For the quantum numbers with  $\eta_{CP} \cdot \tau = -1$ , the contact graph will contribute for special values of the angular momentum ( $j=0$  or  $j=1$ ). Therefore we have no analyticity in  $j$  in the corresponding channels. A similar situation occurs in the scalar  $\varphi^4$ -theory<sup>35</sup>.

Even if we restrict ourselves to the case  $\eta_{CP} \cdot \tau = +1$  we encounter some problem for  $j=0$  because

then we have only two non-vanishing wave-functions whereas we have four for all other  $j$  values. Therefore we have analyticity only in  $\text{Re } j > 0$ <sup>36</sup>.

The computation reveals, however, that the full system (4.21) – (4.24) yields the same physical eigenvalues as the (physically meaningful) truncated system (4.21) (4.23). All additional eigenvalues turn out to be unphysical. So we have no problem in the numerical continuation to  $j = 0$ .

For the quantum numbers  $\eta_{CP} = -1$ ,  $\tau = -1$ , we have a physical particle at  $j = 1$  whereas for  $\eta_{CP} = +1$ ,  $\tau = +1$  the lowest spin particle has  $j = 2$  because of signature. So we chose the  $B$ -trajectory for the numerical computation and calculated it for  $j \geq 0$ ,  $j$  real.

Since the matrix size is enlarged by a factor 4 compared to the  $j = 0$  case, we get a  $48 \times 48$  matrix from the smallest matrix size of Table 1. This size has been used in the computation because of lack of storage. Therefore we may expect only a 25% precision, but one may hope that the error is always in the same direction so that the shape of the trajectory is not distorted too much.

The first step in the calculation was the fixing of the coupling constant. For the values (6.9) or (6.10) no real eigenvalues were found as already in the scalar case. Therefore we raised the coupling constant so that the first excited states for  $j = 1$  had the experimental mass of the  $B$  meson that is  $s = 1.52$ . It was not possible to fit the ground state to this mass.

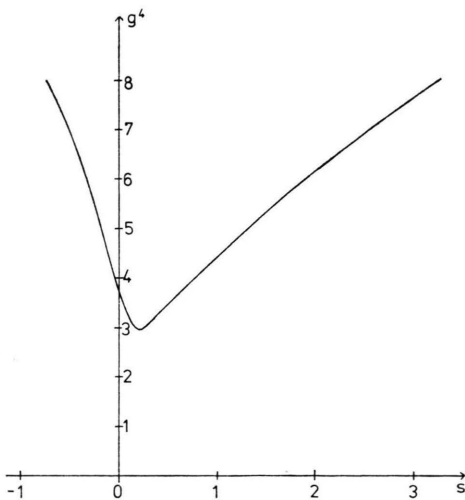


Fig. 5.  $g^4$  dependent on  $s$  for  $B$ -meson. The two lowest states are given. Compare with<sup>31</sup>.

The variation of  $g^4$  with respect to  $s$  is shown in Fig. 5 for  $j = 1$ ,  $\eta_{CP} = -1$  for the ground state and the first excited state. We find the coupling constant

$$g^2 = 2.37. \quad (7.1)$$

With this value for  $g^2$ , we calculated the Regge-trajectory. It is a major advantage of (5.11) that we can fix the coupling constant for the whole trajectory whereas one has to adjust its values in the usual method for each angular momentum.

The resulting trajectory is shown in Figure 6. For  $j > 2.6$  we found no real eigenvalues. We have included the eigenvalue with the correct signature ( $\tau = +1$ ) for  $j = 2$  in order to demonstrate the influence of the exchange term. The shift is not too large, so we have nearly exchange degenerate trajectories.

An agreement with the observed linear trajectories could not have been expected. Therefore we have to rely on a comparison with other calculations for discussing the correctness of the result. Most interesting is the comparison with<sup>37</sup> where the trajectories were computed for a non-relativistic Yukawa potential, also beyond threshold. Figure 6 does not look too different from the figures given there for  $j > 0.6$ .

For  $j < 0.6$  we find a backward moving of the trajectory. This is a special feature of the spinor case. A similar result has been found in<sup>18</sup>, namely that the  $j = 1$  state may be more strongly bound than the  $j = 0$  state in the spinor Bethe-Salpeter equation. So our result might be correct, at least in its principal features.

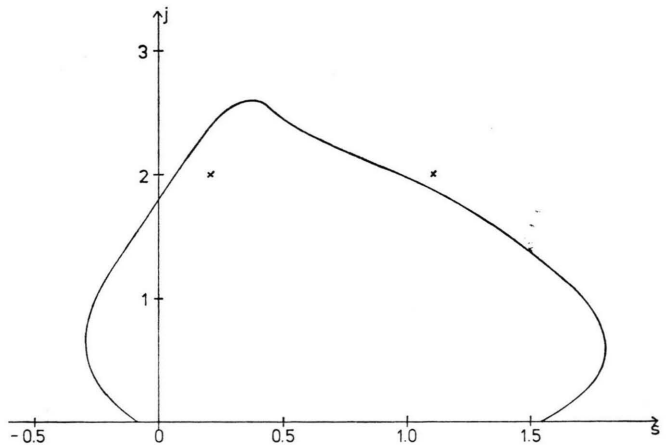


Fig. 6. The  $B$ -trajectory for  $g^2 = 2.37$  (7.1). For  $j = 2$  also the corresponding right signature points are given.

## 8. Conclusions

The results of our work will now be summarized in several points.

- 1) By the variational method one can calculate the eigenvalues of the nonlocal graph with a precision only limited by the size of the computer. Especially it allows the computation of many approximations so that one can judge the error.
- 2) The results for scalar mesons show that the second order is indeed only a correction to the first order. So the approximation scheme is set on a much more solid base. Furthermore, the agreement with the experimental masses is much better than for the first order alone.
- 3) The Fredholm approximation used previously yields completely unreliable results. Hence it should not be used any more.
- 4) In principle, the calculation of meson masses for all quantum numbers is possible, especially for arbitrary angular momentum. This was demonstrated by a calculation of a Regge-trajectory. But before one starts a large scale numerical computation, one should remove some defects of the theory such as parity degeneracy.
- 5) The method is able to include all graphs local in the  $s$ -channel that is those graphs which have an  $s$  independent spectral function. More general graphs will lead to an integro-differential equation.
- 6) Beyond nonlinear spinor theory, we have demonstrated the applicability of the variational method in the spinor Bethe-Salpeter equation. Therefore one may hope to use it in more general kinematical situations such as unequal external masses.

### Acknowledgement

I am indebted to Prof. Dr. H. Stumpf for introducing me to functional nonlinear spinor theory.

## Appendix I

In this appendix we discuss briefly the problems related to the Wick rotation, i. e. to the transformation to the Euclidean region. The discussion is by no means complete, and future work has to be done on this question. A more phenomenological approach may consider (4.4) simply as an approximation to the exact equation in the Euclidean region as it is done in a different context in <sup>28</sup>.

The discussion will be performed in momentum space, the situation is known to be better in coordinate space. So we start with the Fourier transformation of (3.17). For brevity we neglect some numerical factors and the exchange term. Then we get from (3.17)

$$\tilde{\psi}(p) = G_\mu(p + \tfrac{1}{2}I) G_\tau(p - \tfrac{1}{2}I) \int d^4q N_{\lambda\nu}(p-q) \cdot \bar{\sigma}^\mu \sigma^\kappa \bar{\sigma}^\lambda \sigma^\eta \tilde{\psi}(q) \sigma_\kappa \bar{\sigma}^\nu \sigma_\eta \bar{\sigma}^\tau. \quad (\text{I.1})$$

$N_{\lambda\nu}(r)$  is the Feynman integral over the inner loop in (2.19):

$$N_{\lambda\nu}(r) = \int \frac{d^4k}{(2\pi)^4} F_\lambda(r-k) F_\nu(k). \quad (\text{I.2})$$

We do not need its explicit form which is given in <sup>38</sup> but it will suffice to give its spectral representation:

$$N_{\lambda\nu}(r) = i \int_0^\infty \frac{d\mu^2}{r^2 - \mu^2 + i\varepsilon} \varrho_{\lambda\nu}(\mu^2). \quad (\text{I.3})$$

The spectral function is also given in <sup>38</sup>. Hence the nonlocal graph corresponds to an exchange of a continuum of particles with masses starting at  $\mu^2 = 0$ .

(I.1) contains a number of singularities in  $p$ . In their discussion we follow <sup>39</sup>. We have four poles from the external propagators at  $p_0 = \pm \tfrac{1}{2}E + \tfrac{1}{2}|\vec{p}|$  called direct ( $d$ ) poles. Furthermore, we have two branch points ( $e$ ) from the exchanged particle continuum at  $p_0 = q_0 \pm |\vec{p} - \vec{q}|$  with associated cuts running to  $\pm \infty$ . Finally, the function  $\tilde{\psi}(p)$  itself has inelastic cuts ( $i$ ) with branch points at  $p_0 = \mp \tfrac{1}{2}E \pm |\vec{p}|$ .

Most of these singularities are removed by rotating the contour of integration in  $q_0$  to the imaginary axis and continuing  $\tilde{\psi}(p)$  analytically to the imaginary axis in  $p_0$ . In doing so, we will pick up residues from displaced poles and discontinuities from displaced cuts. The analysis <sup>39</sup> shows that we encounter a displaced  $d$  pole and displaced  $i$  cut.

The pole and the cut indicate the possible decay of the meson into two "neutrinos" (the pole) or a higher number of them (the cut). A similar situation is found in the first order as discussed in <sup>1</sup>. There it is shown that one may add an inhomogeneous term to the equation to compensate this possible decay channel. Hence we except that such a mechanism is possible in our case, too. So we will neglect the contributions picked up by the rotation. It should be noted that we will not encounter such displaced

singularities if the bare mass pole in  $G$  is shifted to the physical nucleon mass.

Concerning the singularities not removed by the rotation (the  $dd$ ,  $di$ , and  $ii$  pinch), we do not

worry. They are of no great significance in coordinate space, and the numerical calculations show that one may obtain reliable results in spite of them.

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